POLAR REPRESENTATION OF COMPLEX OCTONIONS

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ABSTRACT: The complex octonions are a non-associative extension of complex quaternions, are used in areas such as quantum physics, classical electrodynamics, the representations of robotic systems, kinematics etc. (Kansu et al., 2012, James et al., 1978). In this paper, we study the complex octonions and their basic properties. We generalize in a natural way De-Moivre’s and Euler’s formulae for division complex octonions algebra.

Key Words: De-Moiver’s formula, Euler’s formula, Complex oction

INTRODUCTION

The octonions are the largest of the four normed division algebras. While somewhat neglected due to their non-associativity, they stand at the crossroads of many interesting fields of mathematics (Baez, 2002). A study of the classical electromagnetism’s energy described by the complex octonions in sixteen dimensions is given by Kansu et al. (Kansu et al., 2012). The complex exponential generalizes to quaternions by replacing i by any unit quaternion µ since any unit pure quaternion is a root of -1. Hence, any quaternion may be represented in the polar form where is a real angle. As with complex numbers and quaternions, any octonion can be written in polar form as where is the absolute value of and is a complex number. In this paper, we introduce the complex octonions algebra, Oc, and study some fundamental algebraic properties of them. The polar representation of complex octonions are given, and then by means of the De-Moivre’s theorem, any powers of these octonions are obtained. Finally, we give some examples for more clarification.

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MATERIAL AND METHOD

A complex octonion \( X \) has an expression of the form
\[
X = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 + A_4 e_4 + A_5 e_5 + A_6 e_6 + A_7 e_7 = A_0 e_0 + \sum_{i=1}^{7} A_i e_i,
\]
(1)
where \( A_0, A_i \) are complex numbers and \( e_i, (0 \leq i \leq 7) \) are octonionic units satisfying the equalities that are given in the table below;

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As a consequence of this definition, a complex octonion \( X \) can be written as
\[
X = x + x',
\]
(2)
where \( x \) and \( x' \), real and pure octonion components, respectively. The set of all complex octonions is denoted by \( \mathbb{O}_c \).

For defined octonion in equation (1), the scalar and vectorial parts can be given, respectively, as
\[
S_x = A_0 e_0, \quad (3)
\]
\[
V_x = A_1 e_1 + A_2 e_2 + A_3 e_3 + A_4 e_4 + A_5 e_5 + A_6 e_6 + A_7 e_7, \quad (4)
\]
A complex octonion \( X \) can also be written as
\[
X = (A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3) + (A_4 + A_5 e_1 + A_6 e_2 + A_7 e_3) e_4 = Q + Q'e, \quad (5)
\]
where \( e^2 = -1 \) and
\[
Q, Q' \in \mathbb{H}_c = \left\{ Q = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 \mid e_1^2 = e_2^2 = e_3^2 = -1, A_0 \in \mathbb{C} \right\}, \quad (6)
\]
the complex quaternion division algebra (Jafari, 2016).
For two complex octonions \( X = \sum_{i=0}^{7} A_i e_i \) and \( Y = \sum_{i=0}^{7} B_i e_i \), the summation and subtraction processes are given as

\[
X \pm Y = \sum_{i=0}^{7} (A_i \pm B_i) e_i.
\] (7)

Addition and subtraction of complex octonions is done by adding and subtracting corresponding terms and hence their coefficients, like quaternions.

The product of two complex octonions \( X = S_x + \vec{V}_x \), \( Y = S_y + \vec{V}_y \) is expressed as

\[
XY = S_x S_y - \left( \vec{V}_x \cdot \vec{V}_y \right) + S_x \vec{V}_y + \vec{V}_x \times \vec{V}_y
\] (8)

Multiplication is distributive over addition, so the product of two octonions can be calculated by summing the product of all the terms, again like quaternions. This product can be described by a matrix-vector product as

\[
XY = \begin{bmatrix}
A_0 & -A_3 & -A_2 & -A_1 & -A_5 & -A_4 & -A_7 & A_6 \\
A_3 & A_0 & -A_1 & -A_2 & A_5 & A_4 & A_7 & -A_6 \\
A_2 & A_1 & A_0 & -A_3 & A_6 & -A_4 & -A_7 & A_5 \\
A_1 & -A_2 & A_3 & A_0 & -A_6 & A_4 & -A_7 & A_5 \\
A_5 & -A_6 & A_7 & A_5 & A_0 & -A_3 & A_1 & A_2 \\
A_6 & A_5 & -A_7 & A_6 & A_0 & A_3 & -A_1 & A_2 \\
A_7 & A_6 & -A_5 & A_7 & A_1 & A_2 & -A_0 & A_3 \\
A_4 & A_3 & A_2 & A_1 & A_0 & A_3 & A_2 & -A_1 & -A_0
\end{bmatrix}
\begin{bmatrix}
B_0 \\
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5 \\
B_6 \\
B_7
\end{bmatrix}
\] (9)

where \( X, Y \in O_c \). Complex octonions multiplication is not associative, since

\[
e_i (e_2 e_4) = e_i e_6 = -e_7,
\]

\[
(e_2 e_4) e_i = e_i e_4 = e_2.
\] (10)

It is clear that subalgebra with bases \( e_0, e_1, e_i, e_j \) \( (2 \leq i, j \leq 7) \) is isomorphic to complex quaternions algebra \( H_c \).

**SOME PROPERTIES OF COMPLEX OCTONIONS**

1) The Hamilton conjugate of \( X = \sum_{i=0}^{7} A_i e_i = S_x + \vec{V}_x \) is

\[
\bar{X} = A_0 e_0 - \sum_{i=0}^{7} A_i e_i = S_x - \vec{V}_x.
\] (11)

The complex conjugate of \( X \) is

\[
X^* = \bar{A}_0 e_0 + \sum_{i=0}^{7} \bar{A}_i e_i = (a_0 - ia_0) e_0 + (a_i - ia_i) e_i + \ldots + (a_7 - ia_7) e_7.
\] (12)

The Hermitian conjugate of \( X \) is

\[
X^* = (\bar{X})^* = \bar{A}_0 e_0 - \sum_{i=0}^{7} \bar{A}_i e_i = (a_0 - ia_0) e_0 - (a_i - ia_i) e_i - \ldots - (a_7 - ia_7) e_7.
\] (13)
It is clear the scalar and vector parts of $X$ is denoted by $s_X = \frac{X + \bar{X}}{2}$ and $v_X = \frac{X - \bar{X}}{2}$.

2) The norm of $X$ is

$$N_X = X\bar{X} = \|X\|^2 = \sum_{i=0}^{7} A_i^2 \in C$$ (14)

If $N_X = 1$, then $X$ is called a unit complex octonion. We will use $O^1_c$ to denote the set of unit complex octonions. If $N_X = 0$, then $X$ is called a null complex octonion.

**Lemma 1.** Let $X, Y \in O_c$. The conjugate and norm of complex octonions satisfy the following properties:

1) $\bar{X} = X$, $(X^*)^* = X$, $(X^\dagger)^\dagger = X$
2) $\bar{XY} = \bar{X}\bar{Y}$, $(XY)^* = Y^*X^*$, $(XY)^\dagger = Y^\dagger X^\dagger$ (15)
3) $\bar{X + Y} = \bar{X} + \bar{Y}$, $(X + Y)^* = X^* + Y^*$, $(X + Y)^\dagger = X^\dagger + Y^\dagger$
4) $N_X = N_{\bar{X}}$, $N_{XY} = N_X N_Y$

3) The inverse of $X$ with $N_X \neq 0$, is

$$X^{-1} = \frac{1}{N_X} \bar{X}.$$ (16)

**Example 1.** Consider the complex octonions

$$X_1 = \frac{\sqrt{3}}{2} + 2e_1 + (1-i)e_2 + 2ie_3 + (1+i)e_4 + i e_5 + \frac{1}{2} e_6 - e_7,$$

$$X_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e_1 + (1-i)e_2 + 2ie_3 + (1+i)e_4 + \sqrt{2} i e_5 + e_6 + 2e_7,$$

$$X_3 = 1 + (2-i)e_1 + ie_2 + e_3 + (1+i)e_4 + i e_5 + e_6 - e_7,$$ (17)

The norms of $X_1, X_2, X_3$ are

$$N_{X_1} = 1, \quad N_{X_2} = 0, \quad N_{X_3} = 5 - 2i.$$ (18)

The conjugates of $X_1, X_2, X_3$ are

$$\bar{X}_1 = \frac{\sqrt{3}}{2} - 2e_1 - (1-i)e_2 - 2ie_3 - (1+i)e_4 - i e_5 - \frac{1}{2} e_6 + e_7,$$

$$\bar{X}_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e_1 + (1+i)e_2 - 2ie_3 + (1-i)e_4 - \sqrt{2} i e_5 + e_6 + 2e_7,$$

$$\bar{X}_3 = 1 - (2+i)e_1 + ie_2 + e_3 - (1-i)e_4 + i e_5 + e_6 - e_7,$$ (19)

The inverse of $X_1, X_3$ are

$$X_1^{-1} = \frac{\sqrt{3}}{2} - 2e_1 - (1-i)e_2 - 2ie_3 - (1+i)e_4 - i e_5 - \frac{1}{2} e_6 + e_7,$$

$$X_3 = \frac{1}{5 - 2i} [1 - (2-i)e_1 - ie_2 - e_3 - (1+i)e_4 - i e_5 - e_6 + e_7],$$ (20)

and $X_2$ not invertible.

**Theorem 1.** The set $O^1_c$ of unit complex octonions is a subgroup of the group $O_c^0$ where $O_c^0 = O_c - \{0-\bar{0}\}$. 
Proof: Let \( X, Y \in O_C \). We have \( N_{xy} = 1 \), i.e. \( XY \in O_C \) and thus the first subgroup requirement is satisfied. Also, by the property
\[
N_x = N_x = N_{x^{-1}} = 1,
\]
the second subgroup requirement \( X^{-1} \in O_C \).

RESULT AND DISCUSSION

Trigonometric Form and De Moivre’s Theorem

Every non-null complex octonion \( X = \sum_{i=0}^{7} A_i e_i \) can be written in the trigonometric (polar) form
\[
X = R(\cos \phi + \bar{W} \sin \phi),
\]
with
\[
R = \sqrt{|N_x|} = \sqrt{\sum_{i=0}^{7} A_i^2}, \quad \cos \phi = \frac{A_0}{\sqrt{|N_x|}} \quad \text{and} \quad \sin \phi = \frac{1}{\sqrt{|N_x|}} \left( \sum_{i=1}^{7} A_i^2 \right)^{1/2}.
\]
The unit complex vector \( \bar{W} = \bar{w} + i \bar{w}^* \) is given by
\[
\bar{W} = (w_1, w_2, \ldots, w_7) = \frac{1}{\left( \sum_{i=1}^{7} A_i^2 \right)^{1/2}} (A_0, A_2, \ldots, A_7).
\]

Example 2. The polar form of the complex octonions \( X_1 = \frac{1}{\sqrt{2}} + (i, 1+i, 2i, 1-i, 2, 0, \frac{\sqrt{3}}{2}) \) is
\[
X_1 = \cos \frac{\pi}{4} + \bar{W}_1 \sin \frac{\pi}{4}
\]
and \( X_2 = i + (1+2i, -1+1, 2-i, -1, 2i, i+1, \sqrt{5}) \) is
\[
X_2 = \cos \phi + \bar{W}_2 \sin \phi, \quad \phi = \cos^{-1} i = \frac{\pi}{2} - i \ln(-1 + \sqrt{5}),
\]
where
\[
\bar{W}_1 = \sqrt{2} (i, 1+i, 2i, 1-i, 2, 0, \frac{\sqrt{3}}{2}) \quad \text{and} \quad \bar{W}_2 = \frac{1}{\sqrt{2}} (1+2i, -i+1, 2-i, -1, 2i, i+1, \sqrt{5}).
\]

It is clear that \( N_{w_1} = N_{w_2} = 1 \) and \( \bar{W}_1 \bar{W}_1 = \bar{W}_2 \bar{W}_2 = -1 \).

Since \( \bar{W}^2 = -1 \) we have a natural generalization of Euler’s formula for generalized quaternions
\[
e^{a\theta} = 1 + \bar{W} \theta - \frac{\theta^2}{2!} - \bar{W} \frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \ldots
\]
\[
= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots + \bar{W} (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)
\]
\[
= \cos \theta + \bar{W} \sin \theta,
\]
for any dual number \( \theta \).
Lemma 2. For every unit vector $\hat{W}$, we have

$$(\cos \theta_1 + \hat{W} \sin \theta_1)(\cos \theta_2 + \hat{W} \sin \theta_2) = \cos(\theta_1 + \theta_2) + \hat{W} \sin(\theta_1 + \theta_2).$$

(28)

Theorem 2. (De-Moivre's formula) Let $X = \sqrt{N_x}(\cos \phi + \hat{W} \sin \phi)$ be a complex octonion. Then for any integer $n$,

$$X^n = (\sqrt{N_x})^n(\cos n\phi + \hat{W} \sin n\phi).$$

(29)

Proof: The proof will be by induction on nonnegative integers $n$ and let $N_x = 1$.

For $n = 2$ and on using the validity of theorem as lemma 1, one can show

$$(\cos \phi + \hat{W} \sin \phi)^2 = \cos 2\phi + \hat{W} \sin 2\phi$$

(30)

Suppose that $(\cos \phi + \hat{W} \sin \phi)^n = \cos n\phi + \hat{W} \sin n\phi$, we aim to show

$$(\cos \phi + \hat{W} \sin \phi)^{n+1} = \cos(n+1)\phi + \hat{W} \sin(n+1)\phi.$$  

(31)

Thus

$$(\cos \phi + \hat{W} \sin \phi)^{n+1} = (\cos \phi + \hat{W} \sin \phi)^n(\cos \phi + \hat{W} \sin \phi)$$

$$= (\cos n\theta + \hat{W} \sin n\theta)(\cos \phi + \hat{W} \sin \phi)$$

$$= \cos(n\phi + \phi) + \hat{W} \sin(n\phi + \phi)$$

$$= \cos((n+1)\phi + \hat{W} \sin((n+1)\phi).$$

The formula holds for all integers $n$:

$$X^{-1} = \cos\phi - \hat{W} \sin\phi,$$

(33)

$$X^{-n} = \cos(-n\phi) + \hat{W} \sin(-n\phi)$$

(34)

$$= \cos n\phi - \hat{W} \sin n\phi.$$

Example 3. Let $X = -\sqrt{3} + (1 + i, 2i, 1 - i, 1, 1, 2)$ be a complex octonion. Every power of this octonion is found with the aid of Theorem 1. For example, $20$-th and $83$-th powers are

$$X^{20} = 2^{20}(\cos 20\frac{5\pi}{6} + \hat{W} \sin 20\frac{5\pi}{6})$$

$$= 2^{20}(-\frac{1}{2} + \hat{W} \frac{\sqrt{3}}{2}) = 2^{19}[-1 + \sqrt{3}(1 + i, 2i, 1 - i, 1, 1, 2)],$$

and

$$X^{83} = 2^{83}(\cos 83\frac{5\pi}{6} + \hat{W} \sin 83\frac{5\pi}{6})$$

$$= -2^{82}(\sqrt{3} + \hat{W}).$$

We investigate some properties of the complex octonions by separating them in two cases:

i) Complex octonions with complex angles ($\phi = \alpha+i\phi$); i.e.

...
\[ X = \sqrt{N_X} (\cos \phi + \bar{W} \sin \phi), \]  
(37)

ii) Complex octonions with real angles \((\phi = \varphi, \varphi^* = 0)\); i.e.

\[ X = \sqrt{N_X} (\cos \varphi + \bar{W} \sin \varphi). \]  
(38)

**Theorem 3.** De Moivre’s formula implies that there are uncountably many unit complex octonion \(X = \cos \varphi + \bar{W} \sin \varphi\) satisfying \(X^n = 1\) for \(n \geq 3\).

**Proof:** For every unit vector \(\bar{W}\), the unit complex octonion

\[ X = \cos \frac{2\pi}{n} + \bar{W} \sin \frac{2\pi}{n}, \]  
(39)

is of order \(n\). For \(n = 1\) or \(n = 2\), the complex octonion \(X\) is independent of \(\bar{W}\).

**Example 4.** \(X = \frac{1}{\sqrt{2}} + (i, 1 + i, 2i, 1 - i, 2, 0, \sqrt{2})\) is of order 8 and \(X = \frac{\sqrt{3}}{2} + (1 + i, 1, 2i, 1 - i, -1, \frac{1}{2}, 2)\) is of order 12.

**Theorem 4.** Let \(X = \cos \varphi + \bar{W} \sin \varphi\) be a unit complex octonion. The equation \(A^n = X\) has \(n\) roots, and they are

\[ A_k = \cos \left(\frac{\varphi + 2k\pi}{n}\right) + \bar{W} \sin \left(\frac{\varphi + 2k\pi}{n}\right), \quad k = 0, 1, 2, ..., n - 1. \]  
(40)

**Proof:** We assume that \(A = \cos \vartheta + \bar{W} \sin \vartheta\) is a root of the equation \(A^n = X\), since the vector parts of \(X\) and \(A\) are the same. From Theorem 2, we have

\[ A^n = \cos n\vartheta + \bar{W} \sin n\vartheta, \]  
(41)

thus, we find

\[ \cos n\vartheta = \cos \varphi, \quad \sin n\vartheta = \sin \varphi, \]

So, the \(n\) roots of \(X\) are

\[ A_k = \cos \left(\frac{\varphi + 2k\pi}{n}\right) + \bar{W} \sin \left(\frac{\varphi + 2k\pi}{n}\right), \quad k = 0, 1, 2, ..., n - 1. \]  
(42)

**Example 5.** Let \(X = \frac{\sqrt{3}}{2} + (1 + 2i, -\sqrt{3}, 2i, i - 1, -1, i - 1, 2) = \cos \frac{\pi}{6} + \bar{W} \sin \frac{\pi}{6}\) be a unit complex octonion.

The cube roots of the octonion \(X\) are

\[ X^{\frac{1}{3}} = \cos \left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \bar{W} \sin \left(\frac{\pi}{6} + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2. \]  
(43)
For $k = 0$, the first root is $X^\frac{1}{2}_0 = \cos \frac{\pi}{18} + \bar{W} \sin \frac{\pi}{18} = 0.98 + 0.17\bar{W}$, and the second one for $k = 1$ is $X^\frac{1}{2}_1 = \cos \frac{13\pi}{18} + \bar{W} \sin \frac{13\pi}{18} = -0.64 + 0.76\bar{W}$ and third one is $X^\frac{1}{2}_2 = \cos \frac{25\pi}{18} + \bar{W} \sin \frac{25\pi}{18} = -0.34 - 0.93\bar{W}$. (44)

Also, it is easy to see that $X^\frac{1}{2}_0 + X^\frac{1}{2}_1 + X^\frac{1}{2}_2 = 0$.

The relation between the powers of complex octonions can be found in the following Theorem.

**Theorem 5.** Let $X$ be a unit complex octonion with the polar form $X = \cos \phi + \bar{W} \sin \phi$.

If $m = \frac{2\pi}{\phi} \in \mathbb{Z} - \{1\}$, then $X^n = X^m$ if and only if $n \equiv m \pmod{p}$.

**Proof:** Let $n \equiv m \pmod{p}$. Then we have $n = a + pm$, where $a \in \mathbb{Z}$.

\[
X^n = \cos n\phi + \bar{W} \sin n\phi = \cos(\frac{2\pi}{\phi} + m)\phi + \bar{W} \sin(\frac{2\pi}{\phi} + m)\phi
\]
\[
= \cos(m\phi + a2\pi) + \bar{W} \sin(m\phi + a2\pi) = \cos m\phi + \bar{W} \sin m\phi = X^m.
\]

Now suppose $X^n = \cos n\phi + \bar{W} \sin n\phi$ and $X^m = \cos m\phi + \bar{W} \sin m\phi$. (47)

If $X^n = X^m$ then we get $\cos n\phi = \cos m\phi$ and $\sin n\phi = \sin m\phi$, which means $n\phi = m\phi + 2\pi a, a \in \mathbb{Z}$. (48)

Thus $n = m + \frac{2\pi}{\phi}a$ or $n \equiv m \pmod{p}$. (49)

**Example 6.** Let $X = \frac{\sqrt{2}}{2} + (1 + 2i, -\sqrt{3}, 2i, i - 1, \frac{1}{\sqrt{2}}, i - 1, 2)$ be a unit complex octonion. From Theorem 5,

\[
m = \frac{2\pi}{\pi/4} = 8, \text{ so we have}
\]
\[ X = X^9 = X^{17} = \ldots \]
\[ X^2 = X^{10} = X^{18} = \ldots \]
\[ X^3 = X^{11} = X^{19} = \ldots \]
\[ X^4 = X^{12} = X^{20} = \ldots = -1 \]
\[ \vdots \]
\[ X^8 = X^{16} = X^{24} = \ldots = 1. \]  

(50)

CONCLUSION

In this paper, we defined and gave some of algebraic properties of complex octonions and investigated the De Moivre’s formulas for these octonions. The relation between the powers of complex octonions is given in Theorem 5. We also showed that the equation \( X^n = 1 \) has uncountably many solutions for any unit complex octonions (Theorem 3).

FUTURE WORK

We will give a complete investigation to real matrix representations of complex octonions, and give any powers of these matrices.

REFERENCES